The Primitive Groups of Class 15.

By W. A. Manning.

There are but three primitive groups of class $3p$, p an odd prime, in which occur substitutions of order p and degree $3p$.* All are of class 15. One is of degree 16 and order 80. The other two are the triply transitive group of degree 17 and order 4080, and its subgroup of order 240. Corresponding to every whole number k greater than 3 there exist four simply transitive primitive groups of class $3k$. The alternating group of degree $k+2$ permutes the $\frac{1}{2}(k+1)(k+2)$ binary products ab, ac, ..., according to a simply transitive primitive group of class $3k$ $(k>3)$.† The three groups ‡

$$(aa'\cdot bb'\cdot \ldots\ldots \cdot kk')(ab\ldots\ldots j) \text{ pos } (a'b'\ldots\ldots j') \text{ pos},$$
$$(aa'\cdot bb'\cdot \ldots\ldots \cdot kk')\{(ab\ldots\ldots j) \text{ all } (a'b'\ldots\ldots j') \text{ all}\}\text{pos},$$
$$(ab'ba'\cdot cc'\cdot dd'\cdot \ldots\ldots \cdot kk')\{(ab\ldots\ldots j) \text{ all } (a'b'\ldots\ldots j') \text{ all}\}\text{pos},$$

are maximal subgroups of

$$(aa'\cdot bb'\cdot \ldots\ldots \cdot kk')(ab\ldots\ldots k) \text{ pos } (a'b'\ldots\ldots k') \text{ pos},$$
$$(aa'\cdot bb'\cdot \ldots\ldots \cdot kk')\{(ab\ldots\ldots k) \text{ all } (a'b'\ldots\ldots k') \text{ all}\}\text{pos},$$
$$(ab'ba'\cdot cc'\cdot dd'\cdot \ldots\ldots \cdot kk')\{(ab\ldots\ldots k) \text{ all } (a'b'\ldots\ldots k') \text{ all}\}\text{pos},$$

respectively. The latter are therefore simply isomorphic to primitive groups of degree k^2, and of order $\frac{1}{2}(k!)^2$, $(k!)^2$, $(k!)^2$, respectively. They are simply transitive and of class $3k$. *When k is* 5, *these three and the alternating group of degree* 7 *written on twenty-one letters are the only primitive groups of class* 15 *that contain no substitution of order* 5 *and degree* 15.

Those primitive groups of class 15 which contain a substitution of order 5 and degree 15 are known. There remain to be determined those in which the substitutions of degree 15 are all of the type

$$s_1 = a_1a_2a_3 \,.\, b_1b_2b_3 \,.\, c_1c_2c_3 \,.\, d_1d_2d_3 \,.\, e_1e_2e_3\,.$$

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* *Transactions of the American Mathematical Society*, Vol. VI (1905), p. 42.

† C. Jordan, *Comptes Rendus*, Vol. LXXV (1872), p. 1754; American Journal of Mathematics, Vol. XXXII (1910), p. 256, where the error in the formula for the class is corrected.

‡ Cayley, *Quarterly Journal of Mathematics*, Vol. XXV (1891), p. 71, where this notation is explained. For the three groups, see Miller, American Journal of Mathematics, Vol. XXI (1899), p. 299, Theorems III and IV.

Since there exists but one group of order 15, the cyclic group, the only other substitution on the same fifteen letters as s_1 is its square. Among all the substitutions in one of our groups G that are similar to s_1, let those substitutions be selected which displace a minimum number of letters new to s_1, s_1^2 excluded. First, it is easy to see that this minimum can not exceed five, the number of cycles in s_1. For if such a substitution s_2 had more than five letters new to s_1, two of them would be found adjacent in one of its cycles. The transform $s_2^2 s_1 s_2$ would then displace a smaller number of letters new to s_1 than the assigned minimum, and this is possible only if $s_2^2 s_1 s_2$ displaces no letters new to s_1, that is, is s_1 or s_1^2. But s_2, a substitution of odd order, can not transform s_1 into its inverse. Then s_1 is invariant in the subgroup H generated by the complete set of conjugates under G to which it belongs, and since one of these conjugates is invariant in H, all are invariant in H and H is Abelian. But H is not a regular group and hence, being Abelian, is not transitive, and can not be invariant in the primitive group G. *Then a substitution* s_2 (*not* s_1^2) *that displaces in its cycles the least possible number of letters new to* s_1, *and is similar to* s_1, *displaces at most five such new letters.*

If s_2 has two of these new letters in one cycle, $s_2^2 s_1 s_2$ leaves one of them fixed, and hence is s_1. Now since s_2 is commutative with s_1, it displaces exactly three new letters, and they form one of the five cycles of s_2. If s_2 does not permute cycles of s_1, either $s_1 s_2^2$ or $s_1 s_2$ is of degree less than 15, and obviously not identity. Then s_2 permutes cyclically three cycles of s_1. If s_2 has not this form, it has at most one new letter to a cycle. Let the group $\{s_1, s_2\}$ when not Abelian be called D. Of every substitution of D it may be remarked that it either permutes the letters new to s_1 only among themselves, or else replaces all of them by letters of s_1. Were this not true a transform of s_1 could at once be found that would displace at least one new letter, and at the same time fewer new letters than s_2. Any substitution of D that replaces one new letter by another new letter transforms s_1 into s_1 or s_1^2. A transitive constituent of D that involves more than one of the letters new to s_1 is imprimitive, and one of its systems of imprimitivity is composed of its new letters. If s_2 is not similar to s_1 in the letters of each of its transitive constituents, $t_1 = s_2^2 s_1 s_2$ is similar to s_1 in those letters and may be used for s_2 in case $\{s_1, t_1\}$ has the same transitive constituents as $\{s_1, s_2\}$. But if t_1 is not similar to s_1 in the new (and smaller) transitive sets we may use $t_2 = t_1^2 s_1 t_1$, $t_3 = t_2^2 s_1 t_2$, or a final transform which is similar to s_1 in the various constituents of the group generated by it and s_1. This apparently fails when t_1, t_2, or one of these transforms is commutative with s_1, but then we may use this Abelian group of

degree 18. *Hence* G *contains a non-Abelian group* D *generated by two similar substitutions* s_1 *and* s_2 *of degree* 15, *which are similar in the letters of each constituent, or an Abelian group* $\{s_1, s_2\}$ *of order* 9 *and degree* 18. In no case do s_1 and s_2 have more than one common cycle, for as many as two common cycles cause the product $s_1s_2^2$ to displace fewer than fifteen letters. It is evident that the number of transitive constituents in D does not exceed 5. If there are exactly five constituents, D is either a simple isomorphism between five tetraedral groups or is a (1, 4) isomorphism between a cyclic group of degree and order 3, and an intransitive tetraedral group of degree 16, which is itself a simple isomorphism between four alternating groups of degree 4. Hence, if s_2 displaces less than four letters new to s_1, it connects cycles of s_1.

If D is of degree 16 or 17 it is transitive or is a simple isomorphism between transitive constituents. If D is of degree 17, 18, or 19, it is intransitive. For if D is transitive, the new letters in s_2 form a system of imprimitivity of D, which is possible only when there are three of them permuted according to a transitive group of degree 6. Since the two generators s_1 and s_2 are of degree 15, neither can do more than permute cyclically three of the six systems. Then they permute different systems and the group in the systems is intransitive, and D is intransitive.

If D is a transitive group of degree 16, it is of order 48 and has an invariant subgroup of degree and order 16. This group can not be contained in a larger group of degree 16, with the present limitations as to the class of G, nor is it contained in any primitive group of degree 17, 18, 19, or 20, if we may rely upon the accuracy of the lists of the primitive groups of these degrees.* If D is a subgroup of a doubly transitive group of degree 21, it must have five systems of imprimitivity of four letters each with a given letter, the new letter x, say, in common.† Now s_1 permutes among themselves the letters of the systems of imprimitivity to which x belongs, and so far as s_1 is concerned x may be a part of as many systems of four letters each as s_1 has cycles, but since s_2 must replace every system involving x by another system, the two letters of s_1 in a cycle of s_2 with x can neither of them belong to any system of which x is a part. Hence D can not be five-fold imprimitive. It can not have systems of imprimitivity of two or eight letters because of its two generators. Then the primitive groups under consideration do not contain a transitive subgroup of degree 16 and order 48. If D is intransitive the constituent involving the new letter x is of degree $3k+1$ and order $3(3k+1)$,

* See AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXV (1913), p. 229, for references.

† *Transactions of the American Mathematical Society*, Vol. VII (1906), p. 500.

$k=1, 2, 3, 4$, but a non-regular constituent of degree 7, 10, or 13 is clearly out of the question.* A tetraedral group may be written as a transitive group on twelve and on six letters. But if D is set up as a simple isomorphism between three tetraedral groups of the degrees 4, 6 and 6, respectively, its class is 12.

There remains the simple isomorphism between two tetraedral groups of the degrees 4 and 12. This is not a subgroup of a primitive group of degree less than 21, of the type we are discussing. There exists in G a substitution s_3 similar to s_1 which directly unites the two transitive constituents of D, and has, if it is assumed that no substitution similar to s_1 having this property displaces fewer letters new to D, in no cycle more than one letter new to D.† Then the group $H=\{D, s_3\}$ is transitive. Its degree does not exceed 21. Every substitution in it replaces a letter of the transitive constituent of degree 12 in D by the same or another letter of that constituent, so that any substitution of H that replaces a letter $y_1, \ldots,$ new to D, by a letter y, certainly fails to juxtapose letters of the two constituents of D. In particular, the subgroup of H that leaves one letter fixed is intransitive. Within this subgroup, which we may call H_1, the subgroup (F) generated by all the substitutions of order 3 and degree 15 is invariant. F is positive and all its constituents are positive. It is known that H is not of degree 17 or 19, nor is H an imprimitive group of degree 16, for then its order is of necessity 48, and this case has been examined. If H is of degree 18, and if the two letters y_1, y_2 do not constitute a system of imprimitivity, the subgroup F is of degree 17, and has two transitive constituents, which may be of the degrees indicated by the partitions 13, 4 and 12, 5. The presence in G of a substitution of order 13 is impossible. Since by hypothesis G contains no substitution of order 5 and degree 15, the partition 12, 5 is not admissible. But H can not have systems of imprimitivity of two letters each because it is generated by substitutions of degree 15 and order 3. If H is of degree 20, systems of imprimitivity of two or of ten letters each are impossible. The five letters left fixed by s_1 can not form a system because s_2 displaces one of them. Hence s_1 does not permute systems of five letters, and it is clear also that s_1 can not permute the letters of each such system among themselves. Hence H is an imprimitive group with five systems of four letters each, which are permuted according to the alternating group of degree 5. Now D, because of its constituent of degree 12, certainly permutes systems, in fact can not fix a system to which a letter y belongs unless the four letters y_1, y_2, y_3, y_4 are one of the systems of imprimi-

* *Transactions of the American Mathematical Society,* Vol. XVI (1915), p. 139.

† *Transactions of the American Mathematical Society,* Vol. XII (1911), p. 375.

tivity of H. That is, there is only one system of imprimitivity in H involving a given letter y_1, so that H, if it is a subgroup of one of our primitive groups, is to be found in a doubly transitive group of degree 21. Now in H_1, the subgroup of H that leaves one letter (y_1) fixed, the letters y_2, y_3, y_4 can be permuted only among themselves, so that any subgroup of H_1 that is similar to D displaces the same letters as D and has the same two sets of letters in its two constituents. If it is assumed distinct from D, the group it and D together generate must coincide with D since one constituent of D is already alternating. Then D is unique in H_1, and the largest subgroup of H in which D is invariant has a transitive constituent of degree 4.* In the doubly transitive group of degree 21, since D is one of only twenty-one conjugates, the largest subgroup in which D is invariant has a doubly transitive constituent of degree 5. But the group of isomorphisms of the tetraedral group is the symmetric-4 group. Then the required doubly transitive group of degree 21 does not exist. Let H be of degree 21. It can not have systems of imprimitivity of seven letters each since no one of its three generators s_1, s_2, s_3 can permute such systems. No letter of D can belong to the same system of three letters with letters y. Hence H is primitive. The intransitive subgroup H_1, fixing one of the five letters y, has no constituent of degree 2,† nor has it a constituent of degree 3.‡ In H_1 there is not a constituent of degree 4 composed of letters y, for then D would be invariant in H_1 and H_1 would have a transitive constituent of degree 12. It is obvious that *in any simply transitive primitive group, the degree of one transitive constituent of a subgroup leaving one letter fixed must exceed that of any transitive constituent of a subgroup leaving two or more letters fixed.* It is also true that *the degree of the subgroup that leaves one letter of a simply transitive primitive group fixed is equal to the degree of any one of its subgroups that is generated by a complete set of its similar substitutions.* F, for example, is of degree 20. Then the possible degrees of the constituents of F are given by the partitions 15, 5 and 16, 4; 13, 7 and 14, 6 being rejected immediately. If the partition is 15, 5, the second constituent is icosaedral, and F is a $(3, 1)$ or a simple isomorphism between its constituents. If F has an invariant head of order 3, the latter cannot be transformed by substitutions of H into any other subgroup of F because every other subgroup of order 3 in F belongs to a set of at least ten subgroups conjugate under the substitutions of H_1. This invariant head of order 3 can not be invariant in

* *Bulletin of the American Mathematical Society*, 2d series, Vol. XIII (1906), p. 20, Theorem I.

† Miller, *Proceedings of the London Mathematical Society*, Vol. XXVIII (1897), p. 536.

‡ Bennett, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXIV (1912), p. 7.

another subgroup leaving one letter fixed, and therefore is conjugate under substitutions of H to another subgroup of order 3 in H_1. But a given subgroup of degree 15 and order 3 is invariant in a subgroup of H that has, in the six letters left fixed by that subgroup, transitive constituents of degree proportional to the number of subgroups in the different sets of conjugates of H_1 which in H are united as part of a single set.* If the isomorphism is simple, D is one of five conjugates and by the same theorem should admit an isomorphism of order 5. Then there remains only the partition 16, 4. Obviously F is a (16, 1) isomorphism between its two constituents, the second of which is alternating. D is a subgroup leaving fixed one of the letters of the larger constituent, so that there are four such subgroups, all conjugate in H_1. Then there is no primitive group of degree 21 as required.

Let the group D be of degree 17. It is not transitive, nor has it five transitive constituents. Because its degree exceeds its class by only two units, D is a simple isomorphism between its constituents. There is no constituent of degree 3. If there is one constituent of degree 4, D is tetraedral, and there must be one, and hence two, constituents of degree 6, which is absurd when D is of degree 17. There can not be a constituent of degree 7. A constituent of degree 8 must be of such a nature that it can be represented as a transitive group on nine letters. This constituent of degree 9 can not be regular, it can not be of class 7, nor can it be of class 8, for then the two generators of order 3 and degree 9 would be found in a regular self-conjugate subgroup of order 9. But its class certainly exceeds 6. The constituent of lowest degree in D can not be of degree greater than 8, so that finally D is not of degree 17.

If D is assumed to be a transitive group of degree 18 it has six systems of imprimitivity of three letters each. Since s_1 (or s_2) can not permute more than three of these systems, D is certainly not transitive. It has been shown above that if s_2 has two letters new to s_1 adjacent in one of its cycles, D is of order 9 with one constituent a transitive group of order 9. It remains to be seen what groups D, if any, are present in G in which the three new letters x occur in three different cycles of s_2. If D has one constituent of degree 3 (a cycle common to s_1 and s_2), there is no other constituent of degree 3, and there corresponds to identity of this constituent a subgroup of order 3 and degree 15 in the other letters, so that D is Abelian of order 9, contrary to the assumption that s_2 distributes its new letters. Then D is a simple isomorphism between its various transitive constituents. If D has one constituent of degree 4, that is, the alternating group of degree 4, there must be a constituent

Bulletin of the American Mathematical Society, loc. cit.

of higher degree, which can only be of degree 6, and then two more constituents of degree 4 each. As a group D, this is entirely possible. If it be assumed that the lowest degree of any constituent is 6, there is just one other constituent, of degree 12 and generated by two substitutions of order 3 and degree 9, and whose order is in consequence 36 or a multiple of 36, and not divisible by 5 because the twelve letters fall into four systems of imprimitivity of three letters each. It follows that the constituent of degree 6 is imprimitive. Since it is generated by two substitutions of order 3 and degree 6 it can not have systems of imprimitivity of three letters each. But the imprimitive groups of degree 6 and order 36 or 72 have systems of three letters each.* No constituent can be of degree 7 (10), when D is of degree less than 21 (20). A constituent of degree 8 requires the presence in D of a constituent of degree 10. This closes the search for possible groups D of degree 18, and we now take up the detailed study of the two groups of the orders 9 and 12.

The next step forward will consist in showing that no one of our groups G contains the Abelian group of order 9 and degree 18 as a subgroup. No confusion need arise if we agree to call this group "D." There are then in D three transitive constituents of degree 3 and one of degree 9. There exists in G, among the substitutions similar to s_1, a substitution s_3 which connects the largest set of D with some other of the sets of D, and which has at most one new letter y in any cycle. Suppose that $H=\{D, s_3\}$ is transitive. If H is of degree 18, it is imprimitive, and has no systems of two, six, or nine letters because of the nature of its generators s_1, s_2, s_3. If H has systems of three letters, the letters x_1, x_2, x_3 of s_2, not in s_1, form a system, and any other system involving x_1 includes also x_2 and x_3. But H can not be a subgroup of a doubly transitive group of degree 19. Nor is H of degree 19. Let H be of degree 20. It is imprimitive. Since the two new letters y_1, y_2 do not by themselves constitute a system of imprimitivity, we can take from H a substitution s that fixes one of the letters y_1, y_2 and displaces the other. The group $\{D, s\}$ separates the letters of the constituents of degree 3 in D from those of the transitive constituent of degree 9. Its order is not divisible by either 5 or 7. Its degree may admit the two partitions, 9, 4, 3, 3 and 9, 4, 6. A constituent of degree 6 is simply isomorphic to the constituent of degree 13. Then there are not two transitive constituents in $\{D, s\}$ of degree 3 each. When the partition is 9, 4, 6, the constituent of degree 4 is alternating, and in (1, 3) isomorphism to the constituent of degree 15. The transitive constituents of degrees 9 and 6 are then each of order 36 and are in simple isomorphism. Now

*Miller, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXI (1899), p. 287.

in any group of degree 6 and order 36, the subgroup of order 9 is invariant, so that the tetraedral quotient group is absurd. Let H be of degree 21. If H has seven systems of imprimitivity of three letters each, the group in the systems is alternating and involves a substitution of order 5, which in the letters of H is of degree 15. Then H is primitive. The following partitions of the degree of F seem possible: 9, 4, 4, 3, 9, 8, 3, 10, 10. When H_1 has a constituent of degree 9, a transform of F, because it includes D, has a transitive constituent in the same letters. If F has two transitive constituents of degree 10, the first is multiply but not quadruply transitive, and since it is positive, it is one of the groups of class 8, of order 360 or 720. The second constituent, also positive, is not primitive because its class is not greater than 6. But the second constituent can not have systems of two or of five letters. Let H be of degree 22. It is primitive. The possible partitions of the degree of F are, after obvious exclusions, 9, 3, 3, 3, 3, 9, 4, 4, 4, 9, 6, 3, 3, 9, 8, 4, 9, 6, 6, 9, 9, 3, 9, 12, 10, 6, 5, 12, 3, 3, 3, 12, 6, 3, 12, 9. In the first five cases H_1 has a transitive constituent of degree 9, and can not be maximal. In the next case H_1 has a transitive constituent of degree 3.* Now consider F when its transitive constituents are of the degrees 10, 6, 5. The constituents of degree 10 and of degree 6 are multiply transitive. The constituent of degree 5 is alternating, and the constituent of degree 6, being a simple group, is in simple isomorphism to it. Then F is a simple isomorphism between three icosaedral groups. But all the substitutions of order 3 in such an intransitive group are of degree 18. For it has been noted before in this paper that the icosaedral group when written as a transitive group on ten letters is of class 8, and when on six letters is of class 4. It is now possible to state that 5 does not divide the order of F. Any transitive constituent of degree 12 is imprimitive. In the last three partitions, the three letters y must form a system of imprimitivity, for otherwise a certain subgroup $\{D, s\}$ of F would have a transitive constituent of degree 10 or 11. The partition 12, 3, 3, 3 is impossible. Likewise 12, 6, 3 is impossible because the constituents of degree 12 and 6 must be simply isomorphic, while 108 does not divide the order of any group of degree 6. If the constituent of degree 9 (when the partition is 12, 9) is primitive its class is 6, and in consequence its order is 216, a number not divisible by 108. Then the second constituent has systems of imprimitivity of three letters each. These systems are permuted by neither s_1 nor s_2, nor by any of their conjugates in F. Now F has at least one substitution similar to s_1 that adds the new system y_1, y_2, y_3 to the first constituent, using three or more

* Bennett, *loc. cit.*

cycles for that purpose, and which consequently does not unite the other sets of D. Such a group was shown to be impossible in the discussion of the partitions 12, 3, 3, 3 and 12, 6, 3. Then the first constituent of F is of degree 9, and the second of degree 12. Now the subgroup F' of H_1' that fixes a second letter y, and which also includes D, has a transitive constituent of degree 9. Since the partition 12, 9 has been shown to be impossible, this constituent of degree 9 in F' is on the letters of the largest transitive constituent of D. Then H_1 is not maximal. Let H be of degree 23.* In this case H_1 is transitive and F has two transitive constituents of degree 11. The first being at least triply transitive is of order 7920 and class 8. But the class of the second constituent does not exceed 6.

The group $H'=\{D, s_3\}$ is intransitive. It will be convenient to have before us the actual substitutions of the group D. They are:

$$
\begin{aligned}
&1,\\
s_1s_2&=a_1b_2c_3\,.\,a_2b_3c_1\,.\,a_3b_1c_2\,.\,d_1d_3d_2\,.\,e_1e_2e_3\,.\,x_1x_2x_3\,,\\
s_1^2s_2^2&=a_1c_3b_2\,.\,a_2c_1b_3\,.\,a_3c_2b_1\,.\,d_1d_2d_3\,.\,e_1e_3e_2\,.\,x_1x_3x_2\,,\\
s_1&=a_1a_2a_3\,.\,b_1b_2b_3\,.\,c_1c_2c_3\,.\,d_1d_2d_3\,.\,e_1e_2e_3\,,\\
s_1^2s_2&=a_1b_3c_2\,.\,a_2b_1c_3\,.\,a_3b_2c_1\,.\,e_1e_3e_2\,.\,x_1x_2x_3\,,\\
s_2^2&=a_1c_1b_1\,.\,a_2c_2b_2\,.\,a_3c_3b_3\,.\,d_1d_3d_2\,.\,x_1x_3x_2\,,\\
s_1^2&=a_1a_3a_2\,.\,b_1b_3b_2\,.\,c_1c_3c_2\,.\,d_1d_3d_2\,.\,e_1e_3e_2\,,\\
s_2&=a_1b_1c_1\,.\,a_2b_2c_2\,.\,a_3b_3c_3\,.\,d_1d_2d_3\,.\,x_1x_2x_3\,,\\
s_1s_2^2&=a_1c_2b_3\,.\,a_2c_3b_1\,.\,a_3c_1b_2\,.\,e_1e_2e_3\,.\,x_1x_3x_2\,.
\end{aligned}
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This may be transformed into $\{s_1,\, a_1b_1c_1\,.\,a_2b_2c_2\,.\,a_3b_3c_3\,.\,d_1d_3d_2\,.\,x_1x_2x_3\}$ by means of $b_1c_1\,.\,b_2c_2\,.\,b_3c_3\,.\,x_2x_3$. The group D is also invariant under all the substitutions of the group

$$\{d_1e_1\,.\,d_2e_2\,.\,d_3e_3\,.\,b_1c_2\,.\,b_2c_3\,.\,b_3c_1\,.\,x_2x_3,\quad d_1x_1\,.\,d_2x_2\,.\,d_3x_3\,.\,a_2b_3\,.\,b_2c_3\,.\,c_2a_3\,.\,e_2e_3,\quad d_1d_2d_3\}.$$

Hence we are at liberty to write $s_3=(a_1d_1\text{—})\ldots\ldots$ One of the substitutions of degree 15 in $s_3^2Ds_3$ has the property of connecting the set $a_1,\ldots,$ of D and another set. Hence for s_3 we may choose a substitution similar in its transitive constituents to one of the substitutions of D. Whenever the transitive constituent $a_1, a_2, \ldots,$ of H' is of degree less than 18, every substitution of H' replaces some one of the nine letters $a_1, a_2, \ldots,$ of D by one of those letters, so that it is evident that a substitution of H' which replaces a y by a y (new letters to D) does not have a letter of the set $a_1, a_2, \ldots,$ of D in any cycle with one of the remaining letters $d_1, \ldots,$ of D. Partitions of the degree of H' not entailing substitutions of order 5 and degree 15, of order 7 and

* Burnside, "Theory of Groups," 2d edition (1911), p. 341.

degree 14, and so forth, are 12, 3, 3, 12, 6, 12, 4, 3, 16, 3, 12, 4, 4, 12, 8, 16, 4, 14, 7, 15, 6, 18, 3, 16, 3, 3, 14, 8, 15, 7, 16, 6, 18, 4, 16, 4, 3, 15, 8, 16, 7, 20, 3. Consider the last case: 20, 3. The constituent of degree 20 is not primitive, nor has it systems of imprimitivity of degree 2 or 10. Only s_3 permutes systems to which the letters y belong. Then there are not systems of four letters each. If s_3 does not already fix x_1, x_2, x_3, transform D and s_3 by the substitution $e_1x_1 \,.\, e_2x_3 \,.\, e_3x_2 \,.\, b_1a_3 \,.\, b_2c_3 \,.\, a_2c_1 \,.\, d_2d_3$ under which D is invariant. Then s_3 fixes x_1, x_2, x_3. Then s_2 does not permute systems, nor consequently, does s_1 if we look at $d_1d_2d_3$, but it does if we look at $a_1b_1c_1$. The partition 20, 3 is impossible. If the partition is 18, 4 or 18, 3 the larger constituent is imprimitive and can only have systems of three letters, which in turn are permuted according to a primitive group of class 3. Here we encounter substitutions of order 5 and degree 15. A transitive constituent of degree 16 in H' involves one or four new letters y. Its order is at least 144. Since G can not contain a subgroup of degree 16 of order greater than 16, the partitions 16, 3 and 16, 4 are impossible. Let the partition be 16, 3, 3. H' is a (16, 1) isomorphism between a transitive group of order 144 and a direct product of order 9. The first constituent is not primitive. It has no systems of two or eight letters, hence its four systems of imprimitivity are permuted according to the alternating group of degree 4. The intransitive head of order 12 is also of order 12 in each transitive constituent, hence is a simple isomorphism between four tetraedral constituents. Now the transitive constituents of degree 3 are in the letters $e_1e_2e_3$, and $x_1x_2x_3$. The substitution $s_1^2s_2$ is in the intransitive head of the constituent of degree 16. It is of degree 9 in those letters whereas it should be of degree 12. Let H' have constituents on sixteen and on six letters. Since the only group of degree 6 whose order is divisible by 144 is the symmetric group, H' is not a simple isomorphism. The constituent of degree 6 is alternating because it is generated by three cycles of degree 3. The constituent of degree 16 is primitive and therefore is of order 5760. There are two such groups, but they are both of class 12 while our constituent in H', because of $s_1^2s_2$, is of class less than 12. If the partition is 16, 4, 3, and if we are not to have a subgroup of degree 16 with the factor 3 in its order, the intransitive constituent of degree 7 is a direct product of order 36. The large constituent is of order 144, 288, or 576, as the isomorphism is (4, 1), (8, 1) or (16, 1). This alone shows that the first constituent is imprimitive when of order 144 or 288. Suppose it is primitive of order 576. Its class should be 12, but $s_1^2s_2$ is of degree 9 in the letters of this constituent. This first constituent because of $s_1^2s_2$ can not have systems of imprimitivity of two letters each. Systems of four

letters each are left unchanged by $s_1^2 s_2$ of degree 9 in the leading constituent, and these four systems are permuted according to the tetraedral group. The intransitive head consisting of all the substitutions of the constituent in question that do not permute systems is then of order 12, 24, or 48. A transitive constituent of this head is of degree 4, and is alternating or symmetric. If the order of the head is 12 or 24 it is a simple isomorphism, and all its substitutions of order 3 are of degree 12, a condition violated by $s_1^2 s_2$. If the order is 48 the symmetric group is not involved. There are only two ways by which we can get the order 48 from four alternating groups of degree 4. In one way an intransitive constituent of degree 12 is formed as a simple isomorphism, and that is put in a (4, 4) isomorphism to a fourth alternating constituent. Another way is to put in (4,4) isomorphism two similar intransitive tetraedral groups on eight letters. But in any event the substitutions of order 3 in the head are of degree 12. The constituent of degree 7 is on the letters $e_1e_2e_3x_1x_2x_3y_5$, and is an alternating group. The alternating group of degree 7 can not be written as a transitive group of degree 16. Then it should be a (16, 1) isomorphism between a primitive group of degree 16 and order 40320 and the alternating group of order 2520. But the former group is of class 12 and does not contain its part of $s_1^2 s_2$. Let H' have a transitive constituent of fifteen letters. Its order is divisible by 135. No non-alternating primitive group of degree 15 has the factor 27 in its order. Then a constituent of degree 15 has five systems of imprimitivity of three letters each, permuted according to an icosaedral group. The smaller transitive constituent, containing a substitution of order 5, is an alternating group or the icosaedral group written on six letters. None of the orders of these groups are divisible by 27, and the last not even by 9. A (3,1) isomorphism between the imprimitive group of degree 15 with its icosaedral quotient group and one of these three alternating groups is impossible. Let H' have a leading constituent of degree 14. It is a simple isomorphism. The constituent of degree 14 is not primitive. It must have seven systems of two letters each. But $s_1^2 s_2$ can not respect such systems if $s_3 = a_1 d_1 — \ldots.$ The partition 12, 3, 3 demands an absurdity, a simple isomorphism between an intransitive constituent of degree 6 and order 9, and a transitive constituent of degree 12. Let 12, 6 be the next partition for study. The second constituent is imprimitive because 5 can not divide the order of H' in this case. But the group generated by $(e_1e_2e_3)$, $(x_1x_2x_3)$ and $(e_1x_1 —)\ldots$, the last of degree 3 or 6 in these letters, is necessarily alternating. Let the partition of the degree of H' be 12, 4, 3. The constituent of degree 7 can not be of order 12, for then the leading constituent is the regular tetraedral group

and all the substitutions of order 3 in H' are of degree 18. Let the intransitive constituent be of order 36, a direct product. It has an invariant subgroup of order 12 and degree 16 which is a simple isomorphism. This subgroup has been fully considered. Let the partition be 12, 4, 4. The group H' is not a simple isomorphism, nor does it involve the direct product of two tetraedral groups. The intransitive constituent of degree 8 is therefore of order 48, and this is the order of H', a number not divisible by 9. The remaining partition is 12, 8. The order of the second constituent of this simple isomorphism is 72 or a multiple of 72. Since it is of class 3, it is not primitive: $\{e_1e_2e_3, x_1x_2x_3\}$ does not respect systems of two letters, and s_3 can not permute systems of four letters.

Let D be the group of degree 18 and order 12 for present consideration. There exists a substitution s_3 similar to s_1 that replaces a letter of the transitive constituent of degree 6 by a letter of another constituent and that has no two new letters, new to D, in one cycle. Let H be transitive if possible. If H is of degree less than 21, it is imprimitive. If H is of degree 18, there is no system of imprimitivity of degree 9, 6, or 2. In case there are systems of three letters, the four letters of a constituent of degree 4 in D belong to different systems, while the constituent of degree 6 has systems of two but not of three letters, so that D can not permute systems of three letters when H is of degree 18. Nor could it do so if H were an imprimitive group of degree 21. If H is imprimitive of degree 20, no systems are of two or ten letters. Systems of five letters are also readily seen to be impossible. For if s_1 permutes systems, y_1, y_2, x_1, x_2, x_3, constitute a system, not respected by s_2. Similarly s_2 can not permute systems. Finally H can not have systems of four letters each, for the reason that no letter of D can belong to a system of four letters with y_1. If H is imprimitive of degree 21, it was noticed above that there can not be systems of three letters, and systems of seven letters are obviously impossible. If H is primitive of degree 21, we can find a substitution s, similar to s_1, in H which fixes one of the letters y_1, y_2, y_3, and displaces at least one of them. This substitution s can not, by hypothesis, connect the constituent of degree 6 of D with another constituent of D. The order of $\{D, s\}$ is not divisible by 5 (the degrees of the transitive constituents of $\{D, s\}$ are not all divisible by 5), 7, or 13. Then if $\{D, s\}$ is of degree 19, the transitive constituents in it are of the degrees indicated by the partition 6, 9, 4, which is an impossibility because the constituent of degree 4 is of order 12, and the intransitive constituent of degree 15 can not be in multiple isomorphism to it. If $\{D, s\}$ is of degree 20, s adds two letters to the constituent of degree 6 of D.

The degrees of the constituents can not be given by 8, 4, 4, 4, because the order of the intransitive constituents of degree 8 is 12, 48, or 144, while that of the other intransitive constituent in simple isomorphism to it is a multiple of 32. If the partition is 8, 8, 4, $\{D, s\}$ is not a simple isomorphism between its three constituents, since one is alternating of degree 4, but the two constituents of degree 8 are in simple isomorphism to each other. The constituent of degree 8 that involves two new letters permutes these two new letters as a system of imprimitivity, one of four, which go according to the alternating group of degree 4, because the constituent in D admits no other systems than those of two letters, and permutes them in a cycle of order 3. The head involves substitutions of degree 4 (as in D), and hence the class of G is lowered. If the degrees of the transitive constituents of $\{D, s\}$ are 8, 12, the above remarks still apply to the first constituent, though the class of G is not necessarily lowered by the presence of substitutions of degree 4 in the head of the constituent of degree 8. The other constituent, in simple isomorphism to the one of degree 8, has the factor 9 in its order because of the substitutions of order 3 and degree 9 in it. This, the smaller constituent, whose order is twelve times a power of 2, does not permit. If H, transitive, is of degree 22, it is primitive, and we may again study the group $\{D, s\}$, confining ourselves to the degree 21, since the preceding remarks in regard to the degrees 19 and 20 apply to this new $\{D, s\}$ without change. To be certain that s may be found without three of the new letters y in one cycle, recall that H, being primitive, has in it a substitution t which replaces a y by a y, and replaces a letter of D by a y. Then $t^{-1}Dt$ has no constituent of degree less than 4, fixes one (and only one) y, and gives us the substitution s similar to s_1 with the three letters y in three cycles. The subgroup F, invariant in H_1, keeps the letters of the constituent of degree 6 of D and the other letters of D in separate constituents. F is of degree 21. If F has a constituent of degree 9, made by adding three letters y to the constituent of degree 6 in D, that constituent is primitive, and s may be chosen so that the degree of $\{D, s\}$ is 19 or 20. The possible partitions of the degree of F, which do not obviously lead to substitutions of order 11, of order 7, of order 5 and degree 15, are 6, 5, 5, 5, 6, 10, 5, 6, 9, 6, 6, 15, 8, 9, 4. We note that in F, and hence also in the other three subgroups that are conjugate to F and leave one of the four new letters y fixed, a transitive constituent that contains the six letters of the largest transitive constituent of D is of degree 6 or 8, thus compelling the presence of a transitive constituent of degree 6 on the same letters in two subgroups, F and one of its transforms. This is impossible when H_1 has a transitive constituent in these six letters, as must be

the case if the partitions are 6, 5, 5, 5, 6, 10, 5, or 6, 15. If the partition is 6, 9, 6, the third constituent is alternating, and G contains a substitution of degree 15 and order 5. If one constituent of F is of degree 4, its order is 12, and the group is not a simple isomorphism between its three transitive constituents, but the constituents of degree 8 and degree 9 are simply isomorphic. The former is imprimitive and its order is not divisible by 9. We conclude that H is not a transitive group of degree 22. If H is of degree 23 it is doubly transitive. The subgroup F of H_1 must have its transitive constituents of equal degree, but the degree of one of them can range only from 6 to 10. Hence H is not of degree 23.

The group $H' = \{D, s_3\}$ is intransitive. Since s_3 replaces one of the six letters of the largest transitive constituent of D by a letter of that constituent, the constituent of degree 6 in $s_3^2Ds_3$ or $s_3Ds_3^2$ has letters of the constituent of degree 6 and of one of the constituents of degree 4 of D in a cycle of one of its generators of order 3. Then it may be assumed that H' has the three generators of any one of its constituents similar substitutions. The possible partitions of the degree of H' are 10, 5, 5, 10, 10, 12, 4, 4, 12, 8, 15, 5, 12, 9, 16, 4, 16, 5, 12, 5, 5, 12, 10, 18, 4, 18, 5. A transitive constituent of degree 12 is not doubly transitive because there is not another constituent of so high a degree as 11. Then it is an imprimitive group generated by three similar substitutions of order 3 and degree 9, whence it follows that it does not admit systems of two, four or six letters. Its systems of three letters go according to the tetraedral group. Then the order of this constituent is divisible by 144, but not by 5, a remark which disposes of the partitions 12, 5, 5 and 12, 10, and which requires that the intransitive constituent on eight letters in the case 12, 4, 4 be a direct product. But the direct product of two alternating groups of degree 4 can not be written as a transitive group of degree 12. When the partition is 12, 8, the second constituent, generated by two substitutions of degree 6 and order 3, has four systems of imprimitivity of two letters each so that its order is not divisible by 9. But in this case the two constituents are in simple isomorphism. A constituent of degree 9 generated by similar substitutions of degree 6 and order 3 is primitive, and being positive and not triply transitive (to avoid a substitution of order 7), is the group of order 216. Again the two constituents are in simple isomorphism. An absurdity has been reached: 144 does not divide 216. Consider the partitions 18, 4 and 18, 5. The larger constituent is not primitive, and since it is generated by four cycles of s_1, s_2, and s_3, it has nine systems of imprimitivity permuted according to a primitive group. The substitutions of order 2 in D are of degree 4 in these

systems, for they do not interchange systems of two letters in the constituent of D of degree 6, and in the constituents of D of degree 4 no two letters of the same constituent can belong to the same system of imprimitivity. Now the group in the nine systems is not alternating, and there is no primitive group of class 4 of degree greater than 8. If the partition is 15, 5, the larger constituent, with the generators s_1, s_2, s_3, can not be imprimitive. If it is primitive it is simply transitive (to avoid a substitution of order 7) and of class greater than 9. There is but one group satisfying these conditions, and it is of order 360, simply isomorphic to the alternating group of degree 6, a simple group. In the case of a transitive constituent of degree 16, systems of imprimitivity of two letters are possible so far as D is concerned. But the two new letters in s_3 can not form part of a larger system of four or eight letters. The systems of two letters are therefore permuted according to a primitive group of degree 8, and again we encounter a substitution of order 7. Let H' have two transitive constituents of degree 10. The second constituent of degree 10 is generated by three substitutions of order 3 and degree 6. If the two new letters of s_3 do not form a system of imprimitivity, H' has a subgroup $\{D, s\}$ (s is similar to s_1 and is determined in the usual way) of degree 19. Among the small number of partitions of which the subgroup $\{D, s\}$ admits only one claims attention: 6, 9, 4. Here the intransitive constituent of $\{D, s\}$ of degree 15 must be in (3, 1) isomorphism to the alternating constituent of degree 4. But the constituent of degree 6 can not have an invariant subgroup of order 3, because of the substitutions of D in those six letters. Then the second constituent of H' has five systems of imprimitivity and its order is sixty times a power of 2. The other constituent is a positive primitive group (see its generators s_1, ...) of order 360 or 720, multiples of 9. The primitive group of degree 10 and order 60 was passed over because it is not simply isomorphic to an imprimitive group of degree 10. But one partition is left: 10, 5, 5. In this case the intransitive constituent of degree 10 can only be a simple isomorphism between two icosaedral groups, because the direct product of two icosaedral groups can not be represented as a transitive group on ten letters. Hence the transitive constituent of degree 10 is also of order 60, a simple isomorphism between three icosaedral groups of the degrees 10, 5, and 5.

Because H' contains a substitution of order 5 and degree 20, the highest degree of any primitive group of which it is a subgroup is 25. Let now the substitution s_4 be chosen subject to the same conditions as was s_3. Let it be further assumed that $H = \{H', s_4\}$ is transitive. H is not a primitive group of degree 20. Since the transitive constituent of degree 10 in H' is primitive,

systems of imprimitivity can only be of two or ten letters each. But such systems can not be permuted by any of the generating substitutions s_1, s_2, s_3, s_4. Let H be of degree 21. These generators show that it is not imprimitive. The subgroup F, generated by all the substitutions $s_1, \ldots,$ in H_1, the subgroup of H that leaves one letter fixed, is by hypothesis intransitive. Now F can not have a constituent of degree 15 unless it includes a substitution similar to s_1 that connects the transitive constituent of degree 10 of H' with another constituent of H' and displaces fewer new letters than s_4. Suppose that F has two constituents, both of degree 10. The icosaedral group of degree 10 (of class 8) is not contained in a larger group of the same degree and class,* nor in a positive group of degree 10 and class 6. Then F coincides with H'. The largest primitive group of degree 10 in which this icosaedral group on ten letters is invariant is of order 120, isomorphic to the symmetric group on five letters. Then the order of H is 1260 or 2520. But consider for a moment the possible order of the largest subgroup of H in which a subgroup of order 7 (and degree 21) is invariant. This subgroup can contain no substitution of order 3 or 2 capable of transforming a substitution of order 7 into a power without permutation of cycles, for all the subgroups of order 3 in H_1 are of degree 15, not 18, and subgroups of order 2 are of degree 16. This is evident when H_1 is of order 60, but perhaps requires explanation if the order of H_1 is 120. In the latter case one constituent of H_1 is, as we have seen, the primitive group of order 120 of class 6; it has fifteen substitutions of order 2 and degree 8, and ten substitutions of order 2 and degree 6, the latter corresponding to the ten transpositions of five letters. The other constituent of H_1 is then not intransitive (the class must be maintained) but is the imprimitive representation of the symmetric-5 on ten letters. This imprimitive constituent has two systems of five letters each, and also five systems of two letters each. The subgroup that leaves one letter fixed, leaves two fixed, and is the alternating group in the four systems. Then the imprimitive constituent has fifteen substitutions of order 2 and degree 8, and ten substitutions of order 2 and degree 10. In H_1, therefore, all the substitutions of order 2 are of degree 16. No substitution of H can transpose two cycles of a substitution of order 7 and leave one cycle unchanged, the class of H being 15. Hence the order of the largest subgroup of H in which a substitution of order 7 is invariant is 7 or 21. Of the four quotients 1260/7, 2520/7, 1260/21, 2520/21, only 120 is congruent to unity, modulo 7. Then H_1 is of order 120, and there are in H 120 conjugate subgroups of order 7, each of which is invariant in a group of order 21. In H there are one hundred and twenty-six conjugate subgroups of order 5. An

* AMERICAN JOURNAL OF MATHEMATICS, Vol. XXXII (1910), p. 254.

invariant subgroup (K) of H is transitive, hence 7 is a divisor of its order, which can only be 840. But it is now clear that K must have in it the one hundred and twenty-six subgroups of order 5, giving to K more than eight hundred and forty operators. Then H is a simple group. It is well known that there is only one simple group of order 2520, the alternating group of degree 7. It has a transitive representation of class 15 on twenty-one letters, most readily set up as the group of the permutations of the twenty-one binary products ab, ac, ..., of seven letters. It is not contained in a larger primitive group of the same degree and of class 15. Since all its substitutions of degree 15 are conjugate it is not maximal in a transitive group of degree 22.* Let H be of degree 22. Its generators show that it is primitive. The subgroup F is of degree 21, and has the same primitive constituent, the alternating-5 group on ten letters, as H'. Since F is a simple isomorphism this is impossible. Let H be of degree 23. It is doubly transitive, and F, invariant in the transitive subgroup H_1, has two transitive constituents of degree 11 which are interchanged by half the substitutions of H_1. The first constituent, in which the primitive group of degree 10 and order 60 is maximal, is quite possible, but we get into difficulties with the second constituent, simply isomorphic to it, which is of class 6 or less because of the subgroup D. Next, let H be of degree 24. Because of the primitive constituent of degree 10 in H', and the generators s_1, ..., of H, H is not imprimitive. The only divisions of the letters of F into transitive sets that do not introduce substitutions of order 5 and degree less than 20, of order 7 and degree less than 21, of order 11 and degree less than 22, and cycles of thirteen letters are 11, 12, 12, 11, 12, 6, 5. The 12, 11 partition is impossible because of the substitutions of degree 6 in the second constituent. The non-alternating group of degree 11 when represented on twelve letters is multiply transitive and can not have substitutions of degree 6 in it, as the partition 11, 12 requires. In the case of the partition 12, 6, 5 the first constituent being primitive involves a substitution of order 11. Then H is not of degree 24. Finally the primitive group G that includes H is of degree 25. The order of G is not divisible by 7 or 11. Then all tentative partitions of the degree of F are seen at a glance to be impossible.

The group $H''=\{H', s_4\}$ may be intransitive. If s_4 is not similar to s_1 in the letters of each of the constituents of H'' (as s_3 was similar to s_1) another substitution from $s_4^2H's_4$ will serve for s_4 and will be similar to s_1 in the two sets of letters. For all subgroups of order 3 are conjugate in H', and *a fortiori* similar in the various sets of letters. Now the degree of the constituent of H'' that involves the primitive group of degree 10 and order 60 is at least 15, and

* *Bulletin of the American Mathematical Society, loc. cit.*

at most 19. It is therefore primitive and of class 12. Since the second constituent is alternating of degree 5 or 6, we have to seek those positive primitive groups of class 12 and of degree 15 and 16 that can be isomorphic to one of these two alternating groups. If the isomorphism is simple, the larger constituent is the simply transitive primitive group of degree 15 and order 360. If the isomorphism is not simple it is a (16, 1) isomorphism between a primitive group of degree 16 and order 960 or 5760, and one of the given alternating groups. If the constituent of degree 16 is multiply transitive, every substitution of its subgroup that leaves one letter fixed replaces a letter of the transitive constituent of degree 10 of H' by a letter of that constituent, and one substitution at least replaces a letter of that constituent by a letter of a second constituent of H'. The transform of H' by this substitution contains a substitution similar to s_1 that has the properties of s_4 with fewer new letters. Then the constituent of degree 16 can only be the simply transitive group of degree 16 and order 960 (it is G_7 in Miller's list of the primitive groups of degree 16*). Then H'' is of degree 21 only. All the subgroups of degree 15 in H'' are conjugate. Now a substitution s_5 is chosen (after the manner of the choosing of s_4) which with H'' generates a transitive group H. Because of the fact that the degree of this group can not exceed 25, and because of the presence of the primitive constituent of degree greater than 14 in H'', H is primitive. If its degree is 21, it may be handled by means of the arguments used when s_1, s_2, s_3, s_4 was assumed to be a primitive group of degree 21, H' being in F and thus determining the simple group of order 2520, if any. If H is of degree 22, F coincides with H''. In H'', that is, in H_1, all the subgroups of degree 15 form just one set of conjugates, while one of them leaves fixed seven letters which therefore form a transitive set in the largest subgroup of H in which that subgroup is invariant, which latter group, however, can contain no substitution of order 7. Let H be of degree 23, a doubly transitive group. Then F should have two transitive constituents of degree 11. Let H be of degree 24. F has two distinct sets of letters. A constituent of degree 15, isomorphic to the alternating group of degree 8, is doubly transitive and a constituent of degree 16, isomorphic to the alternating group of degree 7, is triply transitive. To show the non-existence of the simply transitive primitive group H, we may call attention to the following general theorem, proof of which is appended:

If the degree of a transitive constituent of the subgroup leaving one letter fixed in a simply transitive primitive group exceeds by two (or more) units the

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*Miller, AMERICAN JOURNAL OF MATHEMATICS, Vol. XX (1898), p. 231.

*degree of any other transitive constituent of that subgroup, then the constituent of highest degree is a simply transitive group.**

It may be proved in few words. Let G_1 fix the letter x of the simply transitive primitive group G of order g and degree n, and let $a_1, a_2, \ldots, a_m$ be the letters of the transitive constituent of highest degree in G_1, supposed, if possible, to be multiply transitive.

The order of the subgroup of G_1 that fixes a_1 and a_2 is $g/nm(m-1)$, while the order of any subgroup of G_1 that fixes two letters, one or both of which do not belong to the set $a_1, a_2, \ldots, a_m$ is greater than $g/nm(m-1)$. Now consider the subgroup G' of G that fixes a_1. Since the order of the subgroup of G that fixes a_1, x, and a_i, $i=2, 3, \ldots, m$, is exactly $g/nm(m-1)$, x and a_i must be in transitive constituents of the subgroup that fixes a_1 of degree greater than $m-2$; but there is in the subgroup that fixes a_1 only one such constituent. Then either G_1 is not a maximal subgroup of G or the transitive constituent of highest degree in G_1 is but simply transitive. The theorem is proved.

Let H displace twenty-five letters. Its order is not divisible by 7. Since the larger constituent of F (and therefore of H_1) is simply transitive, the other constituent is an alternating group and necessarily introduces the factor 7 into the order of H.

From now on we shall be concerned only with primitive groups G of class 15 in which no two substitutions of degree 15 have so many as twelve letters in common.

The group D of degree 19 is not transitive. It has no transitive constituent of degree 7, 11, or 15. Then it has one constituent of degree 3, represented by a cycle common to s_1 and s_2. If D has a constituent of degree 4, it is tetraedral, and D is a simple isomorphism between two intransitive constituents of degree 12 and degree 7. It is of order 12 and has five transitive constituents. If D has a transitive constituent of degree 8, a second transitive constituent of degree 8 is in simple isomorphism to the first. Two substitutions of degree 6 and order 3 that generate an imprimitive group of degree 8, such as we evidently have here, limit the group to systems of imprimitivity of two letters. It can not have systems of four letters. The group in the systems is tetraedral. The head is not the identity and is not of higher order than 2. There is one and only one group satisfying these conditions, $\{135 . 246,\ 147 . 238\}$ with the head $(12 . 34 . 56 . 78)$.†

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* Cf. Jordan, *Bulletin de la société mathématique de France*, Vol. I (1873), p. 198.

† Miller, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXI (1899), see pp. 323 and 332.

Let us study this group D of order 24. It has no subgroup of order 12. There is of course in G a substitution s_3 similar to s_1 that unites two constituents of D and has at most one new letter to a cycle. One of the transforms by s_3 of the eight substitutions of order 3 in D certainly unites two transitive constituents of D and may be used for s_3. We are thus assured of the advantage of having s_3 similar to s_1 and s_2 in the constituents of $\{D, s_3\}$ whenever it is not transitive.

Let $H=\{D, s_3\}$ be transitive. It is not of degree 19. If s_3 displaces just one letter new to D, it displaces three or four of the letters of D that are fixed by s_1. H is imprimitive and no systems are possible except of four letters, and one such system consists of the four letters of s_2 that are new to s_1. Three of these letters do not form a cycle of s_3, for then the existence of a substitution similar to s_1 with not more than two letters new to s_1 could be deduced. Since s_3 bears this same relation to any one of the four subgroups of order 3 in D, it would need to be of degree 16 at least. Let H be of degree 21. It is primitive. In F the letters are partitioned thus: 9, 8, 3, 8, 8, 4. The first places an impossible constituent of degree 3 in H_1, and in the second case H_1 is not maximal because D is in two subgroups F and F', and thus determines the letters of one or two constituents of H_1 and H_1'. If H is of degree 22 it is primitive. The partitions of the degree of F are 9, 9, 3 and 9, 8, 4. Here there is no constituent of degree 3 in H_1. The constituent of degree 8 is of order 24 and should be simply isomorphic to the one of degree 9. H is not of degree 23. Let H be of degree 24. Systems of imprimitivity of two, six, eight, or twelve letters are obviously impossible. No system containing y_1 can contain any letter of D other than the letters of the constituent of degree 3 in D. Hence H has no systems of three letters each, and of systems of four letters each, two systems are $y_1e_1e_2e_3$ and $y_2y_3y_4y_5$, which is absurd because s_3 does not permute these two systems. Then H is primitive. If F has a constituent of degree 3 or 4 in the letters y, new to D, the partitions of its degree are 8, 8, 4, 3 and 9, 8, 3, 3. The first is impossible, as then H_1 would have a constituent of degree 3. As above, the constituent of degree 9 should be simply isomorphic to the constituent of degree 8 of order 24. The same remarks apply to the partitions 10, 10, 3, 12, 8, 3, 9, 8, 6 and 10, 8, 5. There is one other partition, 8, 8, 7, but H_1 is not maximal when the partition is 8, 8, 7.

Then $H'=\{D, s_3\}$ is not transitive. The possible partitions are 16, 3, 16, 4, 12, 8, 18, 3, 12, 9, 18, 4, 14, 8, 12, 10, 20, 3, 14, 9, 20, 4, 14, 10. Here we have borne in mind that s_3 is similar to s_1 in each of the two constituents. A constituent of degree 14 is not primitive, but since it is generated by three

substitutions of order 3 and degree 9 it can have no systems of imprimitivity. Likewise a constituent of degree 12 is imprimitive, but does not respect systems of imprimitivity of three letters each (y_1 can not be in such a system), while it too has three generators of order 3 and degree 9. In the remaining partitions the larger constituent is generated by three substitutions of degree 12 and order 3. Assume that a constituent of degree 18 or 20 has systems of imprimitivity of two letters each. Each transposition of the substitution (t) of order 2 of D is a system. Transform t into t', a substitution that involves one or two transpositions new to t. The product tt' is not identity and is of degree 8 at most. Then the partitions 18, 3 and 18, 4 are impossible, and a transitive constituent of degree 20 has five systems of imprimitivity permuted according to a primitive group. A system of imprimitivity not composed of the four letters new to D is built from a system of two letters from one, and a system of two letters from another constituent of D. Then the group in the systems is alternating. Now s_3 leaves fixed the eight letters of two systems of the constituent of degree 20, so that t', the transform by s_3 of the substitution t of order 2 of D has four transpositions in common with t. Hence the product tt', not identity, is of degree 12 at most. If the leading constituent is of degree 16, t is invariant in H'. This is clear when we consider that the constituent in question, of order three times a power of 2, is not primitive and must accept the systems of two letters of the two constituents of D, even when it has systems of four letters each. If H' is of order 48 it has no substitution of order 3 not in D. There remains then only one partition: 16, 4. The eight systems of two letters each are not permuted according to a primitive group. Then systems of four letters are permuted according to the tetraedral group. The four letters of D that are fixed by s_1 form a system of imprimitivity. Similarly each subgroup of order 3 of D fixes the four letters of a system of the constituent of degree 16. Then s_3 must displace at least three letters of D new to s_1, three letters of D not in s_2, and so on, that is to say, s_3 displaces letters of four distinct systems of four letters each in four cycles, which, as a matter of fact, permute only three systems.

It can be shown that if $D' = \{s_1, s'\}$ (where s' is a substitution of degree 15 that displaces four letters new to s_1) has fewer than five transitive constituents, it contains (or is) the subgroup of the type D just discussed, generated by two substitutions s_1 and s_2 similar in each transitive constituent. If D' does not include the latter subgroup, it does, however, contain the subgroup D of order 12 that has five constituents. Among the substitutions of D' that are of degree 15 and apart from those which unite no two cycles of s_1, let t be

a substitution that unites a minimum number of cycles of s_1. Both t^2s_1t and ts_1t^2 are similar to s_1 in the several transitive constituents of $\{s_1, t\}$, and $\{s_1, t^2s_1t\}$ or $\{s_1, ts_1t^2\}$ has the same constituents as $\{s_1, t\}$ unless each has just five constituents, in which case t displaces the six letters of two cycles of s_1. But then $s_1^2ts_1$ is not t or t^2, and displaces at most three letters new to t.

Hence G contains no subgroup D of degree 19 with fewer than five constituents.

Let D be the group of order 12 and degree 19. One transitive constituent is of order 3 on the letters $a_1a_2a_3$, and each of the others is an alternating group of degree 4. We may write $s_2=(b_1b_2b_4)\ldots.$, so that $s_1s_2=(b_1b_4)(b_2b_3)\ldots.$, a substitution of degree 16. Now if $s_2=(c_1c_3c_4)\ldots.$, $s_1s_2=(c_1c_2c_4)\ldots.$ Therefore,

$$s_2=a_1a_3a_2 \,.\, b_1b_2b_4 \,.\, c_1c_2c_4 \,.\, d_1d_2d_4 \,.\, e_1e_2e_4 \,.$$

There exists in G a substitution s_3, similar to s_1, which replaces a letter of the set a by a letter of the set b, and by means of appropriate transformations we may write uniquely $s_3=(a_1b_1—)\ldots..$ Write

$$t=s_1^2s_2s_1=a_1a_3a_2 \,.\, b_2b_3b_4 \,.\, c_2c_3c_4 \,.\, d_2d_3d_4 \,.\, e_2e_3e_4 \,.$$

Our substitution s_3 introduces a minimum number of new letters to D. Assume that s_3 displaces the three letters a_1, a_2, a_3, and in three different cycles. Then $t^2s_3t=(a_3b_1—)\ldots.$, displaces at most four letters new to s_3, and yet connects two cycles of s_3. Then s_3 certainly replaces an a by an a. The transform of any one of the eight substitutions of order 3 in D replaces an a by b_1, and can not displace fewer letters new to D than s_3 itself. Hence new letters y in s_3 can follow only a_2 or a_3. This permits s_3 at most one new letter y. Of course, since s_3 unites the cycles $a_1a_2a_3$ and $b_1b_2b_3$ of s_1, it must displace one new letter y. But now, on running over the substitutions of order 3 in D, it is evident that s_3 must displace $yb_4c_4d_4e_4b_2c_2d_2e_2b_3c_3d_3e_3$, and two of the letters $c_1d_1e_1$ in addition to a_1 and b_1, which is absurd. Hence this strong condition, as holding good in all that follows:

No two substitutions of degree 15 *in* G *have more than ten letters in common.*

Let $D=\{s_1, s_2\}$ be of degree 20. First assume it transitive. The five new letters in s_2 constitute one of four systems of imprimitivity permuted according to a tetraedral group. D has no systems of two, four, or ten letters. A system containing one new letter x_1 must contain just one other new letter x_2, or else all the five letters $x_1x_2x_3x_4x_5$. Hence there is one and only one system of imprimitivity to which a given letter x_1 may belong. Then D is to be looked for as a subgroup of a doubly transitive group G of degree 21. All the substitutions of degree 15 in D permute three of the four systems, and

leave fixed each of the letters of one system. The doubly transitive group G contains a substitution s of degree 15 involving a letter y new to D. But s must displace, besides y, four letters from each of the four systems of imprimitivity of D. This is absurd.

Then D of degree 20 is intransitive. Constituents of degree 4 are alternating groups and transitive groups of higher degree are imprimitive groups generated by two similar substitutions of order 3 and whose systems are permuted according to an alternating group. A constituent of degree 8 is the group $\{135.246,\ 147.238\}$ of order 24. The order of a transitive constituent of degree 12 is divisible by 9. Hence the partitions 8, 4, 4, 4, 12, 4, 4 and 12, 8 are out of the question. There are three other possible partitions of the degree of D: 4, 4, 4, 4, 4, 8, 8, 4, 16, 4. A transitive constituent of degree 16 is of order 48 or more. The subgroup that corresponds to identity of the other constituent contains no substitution of order 3, and its order is therefore a power of 2. This leading constituent has substitutions of degree 12 and order 3, and since its order is not divisible by 9, it has no substitution of degree 15. Therefore D, whether the partition is 4, 4, 4, 4, 4, 8, 8, 4, or 16, 4, has exactly four conjugate subgroups of order 3. Each of them leaves fixed five letters which are displaced by the other three. We may now remark that G contains no substitution of degree 15 (not a substitution of D) that displaces fewer than two letters new to D. A fact of which considerable use will be made in what follows is this: if a substitution (t) similar to s_1 displaces or fixes the three letters of one cycle of a substitution (s) similar to s_1, s^2ts displaces at most four letters new to t and is therefore t itself, that is, s and t are commutative. The transitive group H generated by $D, s_3, \ldots.,$ can not be of degree 22. For such a group is primitive and its subgroup F contains substitutions of degree 15 displacing one letter new to D. Nor can H be of degree 23.

Now assume that D is that group which has a transitive constituent of degree 16. If H is of degree 24 it is primitive, and since H_1 can have no constituent of degree 16, only one partition of the degree of F need be proposed, 18, 5. A constituent of degree 18 is imprimitive. But D does not respect systems of three letters, as the substitutions of order 5 require. Let H be of degree 25, and let it first be assumed that H is primitive. Of all the possible partitions of the degree of F only 18, 6 and 20, 4 seem to require more than a glance. In the second partition H_1 has an invariant subgroup of degree 20 that corresponds to the identical substitution in the constituent of degree 4, an impossible combination.* Since a constituent of degree 6 is

* Bennett, *loc. cit.*

alternating, there are as above substitutions of order 5 and degree 15 in the imprimitive constituent of degree 18. Suppose the group H to be imprimitive. A system of five letters can be chosen in but one way if it contains a given letter, and hence H is not invariant in a primitive group of degree 25, nor need we pass the degree 26 in a search for a doubly transitive group of which H is a subgroup. If H is included in a primitive group of degree 25, the totality of substitutions of degree 15 in that group generate a primitive group which must be simply transitive and whose non-existence can be shown by the arguments used when H was assumed to be primitive. The only remaining hypothesis is that H is a subgroup of a doubly transitive group G whose subgroup G_1 is imprimitive. The invariant subgroup F of G_1, generated by all substitutions of degree 15 is imprimitive, having five systems of imprimitivity of five letters each permuted according to the icosaedral group. Its order is 1200 or a multiple thereof. The head of F is of order 20 at least, and from the nature of the group D, which leaves fixed each of the five letters of the system formed of the letters of F that are new to D, the head is not a simple isomorphism between its five constituents. If the head contained a substitution of order 3, each constituent would be an alternating or symmetric group. Alternating constituents, if the class of G is to be maintained, can only stand in simple isomorphism to each other. Then the head contains no substitution of order 3. All the subgroups of degree 15 in G_1 are conjugate under the substitutions of G_1, which implies that the largest subgroup of G in which a given subgroup of degree 15 is invariant has a transitive constituent of degree 11. But G (of degree 26) can have in it no substitution of order 11. Thus the partition 16, 4 of the degree of D is shown to be impossible.

The group D has either five constituents of degree 4, or two transitive constituents of degree 8 and one of degree 4. Now assume that D is the latter group, of order 24. Let s_3, with D, generate a transitive group. The first degree to study is 24. If $H=\{D, s_3\}$ is imprimitive, the four letters new to D form a system; and the four letters of the small constituent of D form a system, by which s_3 or s_3^2 replaces the former system (of new letters). Then s_3 displaces the three letters of a cycle of s_1 and is not commutative with s_1, hence displaces at most four letters new to s_1. Then H is primitive. The partitions of 23, corresponding to the subgroup F are 8, 8, 4, 3, 8, 8, 7, 9, 8, 6, 10, 8, 5, 10, 9, 4, 9, 9, 5. The first puts a constituent of degree 3 in H_1, and the last two involve a substitution of order 5 and degree 15, as do the second and third which have an alternating constituent. If the partition is 10, 8, 5, the constituent of degree 8 must include a substitution of order 5, and therefore affords a substitution of order 7 and degree 14. Let H be of degree 25.

Assume H to be primitive. The partitions of F are 8, 8, 4, 4, 8, 8, 8, 8, 8, 5, 3, 9, 8, 4, 3, 9, 9, 6, 10, 8, 6, 10, 9, 5, 10, 10, 4, 12, 8, 4. The partitions 8, 8, 5, 3, 9, 8, 4, 3, and 10, 10, 4 impose a constituent of three or four letters on H_1 under impossible conditions; a constituent of degree 6 contains a substitution of order 5, thus disposing of 9, 9, 6; while the substitution of order 5 in the constituent of degree 8 (partition 10, 8, 6), and in the constituent of degree 9 (partition 10, 9, 5), leads to a substitution of order 7 in these two cases. A transitive constituent of degree 12 is imprimitive and the six systems of two letters each are permuted according to the alternating group of degree 6. This consideration reveals a substitution of order 5 and degree 15 in F. There remain two partitions, 8, 8, 4, 4, and 8, 8, 8. No constituent of degree 8 is primitive. The first two transitive constituents of F admit only those systems of imprimitivity of two letters each, which are determined by the subgroup D, that is, by constituents of D such as $\{123 . 456, 157 . 428\}$. Such a constituent contains no substitution of order 3 not of degree 6. A generator of F, not in D, can not fix all the letters of one of the constituents of D, for then that substitution would be commutative with s_1 and s_2. Then the generating substitutions of degree 15 of F have just four cycles involving letters of these first two constituents, and the remaining constituent (constituents) is (are) generated by cycles of three letters. Then the third constituent is not a transitive group of degree 8. It is the direct product of two alternating groups of degree 4. A transitive constituent of degree 8, being positive, is of order 24 or 96. Either of these groups may be generated by its substitutions of order 3, so that the intransitive constituent of F of degree 16 could have the factor 9 in its order only if it were the direct product of its two transitive constituents, which clearly is not possible. Let it now be assumed that H, of degree 25, is imprimitive. The group in the systems is the alternating group on five letters. There is only one possible system that contains one of the letters fixed by D, the system constituted of these five new letters. Then at once we know that neither H nor a larger imprimitive group of the same degree which includes H is invariant in a primitive group of degree 25, and that a primitive group of degree n that includes H is $n-24$ times transitive. A primitive group of degree 25, generated by substitutions of degree 15 would be disposed of by the arguments given above. We are only interested in the possible existence of a doubly transitive group G of degree 26. Consider the subgroup F of G_1, generated by substitutions of degree 15. The five systems of imprimitivity of F are permuted according to an icosaedral group, and if the order of F is divisible by 9, there is a substitution of order 3 in the intransitive head that permutes no systems. One of these substitutions of order 3 in the head is of

degree 15, connects two transitive constituents of D and has at most three letters new to D, because a system of imprimitivity of F can be made by two letters from each transitive constituent of degree 8, and one letter from the constituent of degree 4 or by the five letters new to D. Then a subgroup of order 3 is a Sylow subgroup of G_1, which implies a substitution of order 11 in G.

The subgroup $H' = \{D, s_3\}$ is intransitive. We recall that s_3 has at least two letters new to D. Among the eight substitutions of order 3 in D, a given letter a_1 of a constituent of degree 8 is in the same cycle with six of the seven remaining letters of that constituent. Therefore the transform by s_3 of at least one of them will, like s_3, connect two constituents of D in one of its cycles, and may be used for s_3. Hence it may be assumed that s_3 is similar in each of the constituents of H' to one of the substitutions of order 3 in D. The plausible partitions are 18, 4, 18, 5, 20, 4, 20, 5, 10, 12, 8, 14, 9, 14, 8, 15, 10, 14, 9, 15, 10, 15. The partitions involving constituents of degree 14, 18 and 20 yield to the arguments of page 301. A constituent of degree 12 is not primitive, but must admit four systems of imprimitivity of three letters each, from which it follows that a constituent of degree 10 is not simply isomorphic to it. The remaining partitions, 8, 15, 9, 15, 10, 15 should also be simple isomorphisms between the two constituents. Let a constituent of degree 15 be primitive. It is not multiply transitive. Then being of class 8 it is the primitive group of degree 15 and order 720, simply isomorphic to the symmetric group of that order. But this symmetric group can not be written transitively on eight, nine, or ten letters. Then a constituent of degree 15 is imprimitive, and since it is generated by substitutions of degree 9 and order 3, it has five systems of three letters each permuted according to the alternating group of degree 5. Since the second constituents are alternating groups, the partitions 8, 15 and 9, 15 are impossible. A constituent of degree 10 is imprimitive and has systems of two letters permuted according to the alternating group of order 60. Its order is not divisible by 9.

Now consider any two substitutions of degree 15, s_1 and s. If s displaces more than five letters new to s_1 and is not commutative with s_1, $s^2 s_1 s$ displaces exactly five letters new to s_1 and, as has just been shown, does not connect two cycles of s_1. Then

$$s = (a_1 x_1 \text{—})\,(b_1 x_2 \text{—})\,(c_1 x_3 \text{—})\,(d_1 x_4 \text{—})\,(e_1 x_5 x_6),$$

say. If $s = (b_1 x_2 a_2) \ldots .$, $s^2 s_1 s = (x_1 b_1 b) \ldots .$, but s can not replace a_3 by a b. Hence s does not connect two cycles of s_1. By hypothesis the transitive constituents of $\{s_1, s\}$ are in part icosaedral. If there are one or more tetraedral constituents $\{s_1, s\}$ is a direct product and of class less than 15. *Hence if* s

has just two new letters in one of its cycles, it has two new letters in every cycle, and $\{s, s_1\}$ *is a simple isomorphism between five alternating groups of order* 60. *If any two substitutions of degree* 15 *are so related that one connects cycles of the other they are commutative, and have at most nine letters in common.*

The last of the groups D is

$$\{s_1,\ s_2=a_1a_2a_4 \,.\, b_1b_2b_4 \,.\, c_1c_2c_4 \,.\, d_1d_2d_4 \,.\, e_1e_2e_4\}.$$

A substitution s_3 exists that connects two of the constituents of D, and it may be taken uniquely as

$$s_3=a_1b_1c_1 \,.\, a_2b_2c_2 \,.\, a_3b_3c_3 \,.\, a_4b_4c_4 \,.\, x_1x_2x_3 .$$

Next, s_4 may be chosen with at most one letter new to $\{s_1, s_2, s_3\}$ in any cycle. Then there are just two distinct forms that s_4 may assume:

$$s_4=a_1d_1b_1 \,.\, a_2d_2b_2 \,.\, a_3d_3b_3 \,.\, a_4d_4b_4 \,.\, x_1y_1x_2 ,$$

and

$$s_4=a_1x_1a_2 \,.\, b_1x_2b_2 \,.\, c_1x_3c_2 \,.\, d_1y_1d_2 \,.\, e_1y_2e_2 .$$

If we proceed with the second form of s_4, we are led at once to a substitution $s_5=a_1d— \ldots .$, so that we are at liberty to use for s_4 the first only, and quite similarly we have for the next step

$$s_5=a_1e_1b_1 \,.\, a_2e_2b_2 \,.\, a_3e_3b_3 \,.\, a_4e_4b_4 \,.\, x_1z_1x_2 ,$$

and then finally

$$s_6=a_1x_1a_2 \,.\, b_1x_2b_2 \,.\, c_1x_3c_2 \,.\, d_1y_1d_2 \,.\, e_1z_1e_2$$

closes the series of generators of a transitive group H of degree 25 and order 360: the direct product of the two icosaedral groups

$$\{a_1a_2a_3 \,.\, b_1b_2b_3 \,.\, c_1c_2c_3 \,.\, d_1d_2d_3 \,.\, e_1e_2e_3,\quad a_1a_2a_4 \,.\, b_1b_2b_4 \,.\, c_1c_2c_4 \,.\, d_1d_2d_4 \,.\, e_1e_2e_4,$$
$$a_1a_2x_1 \,.\, b_1b_2x_2 \,.\, c_1c_2x_3 \,.\, d_1d_2y_1 \,.\, e_1e_2z_1\}$$

and

$$\{a_1b_1c_1 \,.\, a_2b_2c_2 \,.\, a_3b_3c_3 \,.\, a_4b_4c_4 \,.\, x_1x_2x_3,\quad a_1b_1d_1 \,.\, a_2b_2d_2 \,.\, a_3b_3d_3 \,.\, a_4b_4d_4 \,.\, x_1x_2y_1,$$
$$a_1b_1e_1 \,.\, a_2b_2e_2 \,.\, a_3b_3e_3 \,.\, a_4b_4e_4 \,.\, x_1x_2z_1\}.$$

The primitive group G can contain no other substitution of degree 15, so that H is an invariant subgroup, and *the degree of no primitive group of class* 15 *exceeds* 25. Now instead of the special case before us let us turn to the imprimitive group H' generated by

$$\begin{aligned}
s_2&=a_1a_2 \,.\, b_1b_2 \,.\, c_1c_2 \ldots .\ j_1j_2 \,.\, k_1k_2 ,\\
s_3&=a_1a_3 \,.\, b_1b_3 \,.\, c_1c_3 \ldots .\ j_1j_3 \,.\, k_1k_3 ,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots ,\\
s_k&=a_1a_k \,.\, b_1b_k \,.\, c_1c_k \ldots .\ j_1j_k \,.\, k_1k_k ,\\
s_2'&=a_1b_1 \,.\, a_2b_2 \,.\, a_3b_3 \ldots .\ a_jb_j \,.\, a_kb_k ,\\
s_3'&=a_1c_1 \,.\, a_2c_2 \,.\, a_3c_3 \ldots .\ a_jc_j \,.\, a_kc_k ,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots ,\\
s_k'&=a_1k_1 \,.\, a_2k_2 \,.\, a_3k_3 \ldots .\ a_jk_j \,.\, a_kk_k ,
\end{aligned}$$

in which H (after obvious changes in the lettering) is a characteristic subgroup. H' is of order $(k!)^2$. A primitive group of class $2k$ is generated by H' and the following substitution:

$$\begin{aligned} t = (a_1)\,(a_2b_1)\,(a_3c_1)\,(a_4d_1) \ldots . (a_jj_1)\,(a_kk_1), \\ (b_2)\,(b_3c_2)\,(b_4d_2) \ldots . (b_jj_2)\,(b_kk_2), \\ (c_3)\,(c_4d_3) \ldots . (c_jj_3)\,(c_kk_3), \\ \ldots\ldots\ldots\ldots\ldots\ldots, \\ (j_j)\,(j_kk_j), \\ (k_k). \end{aligned}$$

This group is of order $2(k!)^2$ and is simply isomorphic to the intransitive group

$$(aa' \cdot bb' \cdot \ldots . \cdot kk')\,(abc \ldots . k) \text{ all } (a'b'c' \ldots . k') \text{ all}.$$

It is the largest primitive group of degree k^2 in which H' is invariant.* The transitive subgroups of this group in which

$$(abc \ldots . k) \text{ pos } (a'b'c' \ldots . k') \text{ pos}$$

is invariant are

$$\begin{aligned} &(aa' \cdot bb' \cdot cc' \cdot \ldots . \cdot kk')\,(abc \ldots . k) \text{ pos } (a'b'c' \ldots . k') \text{ pos}, \\ &(aa' \cdot bb' \cdot cc' \cdot \ldots . \cdot kk')\,\{(abc \ldots . k) \text{ all } (a'b'c' \ldots . k') \text{ all}\} \text{ pos}, \\ &(ab'ba' \cdot cc' \cdot dd' \cdot \ldots . \cdot kk')\,\{(abc \ldots . k) \text{ all } (a'b'c' \ldots . k') \text{ all}\} \text{ pos}, \end{aligned}$$

of the orders $(k!)^2/2$, $(k!)^2$, $(k!)^2$, respectively.† Maximal subgroups of these three groups are

$$\begin{aligned} &(aa' \cdot bb' \cdot cc' \cdot \ldots . \cdot kk')\,(abc \ldots . j) \text{ pos } (a'b'c' \ldots . j') \text{ pos}, \\ &(aa' \cdot bb' \cdot cc' \cdot \ldots . \cdot kk')\,\{(abc \ldots . j) \text{ all } (a'b'c' \ldots . j') \text{ all}\} \text{ pos}, \\ &(ab'ba' \cdot cc' \cdot dd' \cdot \ldots . \cdot kk')\,\{(abc \ldots . j) \text{ all } (a'b'c' \ldots . j') \text{ all}\} \text{ pos}, \end{aligned}$$

respectively, with respect to which each can be represented as a primitive group of degree k^2, when k is greater than 3. The first, of order $(k!)^2/2$, is generated by H and t; the second, of order $(k!)^2$, is generated by H, t and

$$s_2s_2' = (a_1b_2)\,(a_2b_1)\,(c_1c_2) \ldots . (k_1k_2)\,(a_3b_3) \ldots . (a_kb_k).$$

It contains the preceding group of order $(k!)^2/2$. The third group, also of order $(k!)^2$, is generated by H, s_2s_2' and

$$\begin{aligned} s_2t = (a_1b_1b_2a_2)\,(c_1b_3c_2a_3)\,(d_1b_4d_2a_4) \ldots . (k_1b_kk_2a_k), \\ (c_3)\,(c_4d_3) \ldots\ldots\ldots . (c_jj_3)\,(c_kk_3), \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots, \\ (j_j)\,(j_kk_j), \\ (k_k). \end{aligned}$$

* AMERICAN JOURNAL OF MATHEMATICS, Vol. XXVIII (1906), p. 236.

† Miller, AMERICAN JOURNAL OF MATHEMATICS, Vol. XXI (1899), p. 299, Theorems III and IV.

It does not contain the first of the three as a subgroup. These groups are simply transitive and are of class $3k$, when k is greater than 3. This last statement can readily be proved in connection with the important proposition that the group $\{H', t\}$ is the largest group on the same k^2 letters in which H is invariant. Both statements are known to be true when k is 4.* Let G' be the largest group of degree k^2 in which H is invariant. It is a simply transitive primitive group because systems of imprimitivity of H with one letter in common can be chosen in only two ways. Its subgroup leaving fixed the letter k_k is intransitive, and has the same two sets of intransitivity as $\{s_2, \ldots., s_j, s'_2, \ldots., s'_j, t\}$, the subgroup of $\{H', t\}$ that leaves k_k fixed. The larger con-constituent of the latter subgroup is the primitive group of degree $(k-1)^2$ belonging to the family $\{H', t\}$. Since $\{H', t\}$ when k is 4 is not contained in a larger group in which H is invariant, the group G' coincides with $\{H', t\}$ for all values of k greater than 3. The smaller constituent in the subgroup G'_1 that leaves k_k fixed is the imprimitive group

$$(a_k k_1 \cdot b_k k_2 \cdot \ldots . \cdot j_k k_j)\,(a_k b_k \ldots . j_k) \text{ all } (k_1 k_2 \ldots . k_j) \text{ all}$$

of degree $2(k-1)$ and order $2[(k-1)!]^2$; all the substitutions in its tail are of degree $2(k-1)$. Those substitutions of G'_1 that are in the division of G'_1 that involves only positive substitutions of

$$(a_k b_k \ldots . j_k) \text{ all } (k_1 k_2 \ldots . k_j) \text{ all}$$

are of degree $3k$ at least, if they are of degree $3(k-1)$ or more in the letters of the other constituent of G'_1. The substitutions of G'_1 which are in the tail with respect to the direct product of the two symmetric groups of degree $k-1$ are of degree $5(k-1)$ at least, if they are of degree $3(k-1)$ or more in the letters of the constituent of degree $(k-1)^2$. Then by a complete induction from the groups of degree 16, the three primitive groups above are of class $3k$. What intransitive subgroups of $\{H', t\}$ contain H, when this group is written as a transitive group of degree $2k$? Only

$$\{(abc \ldots . k) \text{ all } (a'b'c' \ldots . k') \text{ all}\}\text{pos},$$
$$(abc \ldots . k) \text{ all } (a'b'c' \ldots . k') \text{ pos},$$

and

$$(abc \ldots . k) \text{ pos } (a'b'c' \ldots . k') \text{ all},$$

include

$$(abc \ldots . k) \text{ pos } (a'b'c' \ldots . k') \text{ pos},$$

and they are imprimitive when represented on k^2 letters.

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* Miller, "On the Primitive Substitution Groups of Degree 16," AMERICAN JOURNAL OF MATHEMATICS, Vol. XX (1898), p. 234. It may be noted that G_{20} does not contain G_{17} as a self-conjugate subgroup.

In my paper on the groups of class 12 it is stated that a certain doubly transitive group of degree 28, as well as its maximal subgroup of degree 27, seem to be new. As a matter of fact the generators there given correspond respectively to the substitutions $[1,0,0,0,0,0]$, $[0,1,0,0,0,0]$, $[1,1,1,1,1,1]$, $[0,0,0,0,1,0]$, $[0,1,0,0,0,1]$, $[0,0,1,0,1,1]$, $[0,0,1,1,1,1]$, as is readily seen if we put a_1 for the letter $(0,0,1,1,0,0)$. Although Jordan gives this group and its sixty-three substitutions of order 2 and degree 12 in the *Traité* of 1870, page 229, he omits it and its subgroup of order 51840 from the list of primitive groups of the first thirteen classes published in 1872.* My existence proofs on pages 258 and 259 of the paper on the "Groups of Class 12" are faulty. The present identifications are designed to replace them. The substitutions $s_1, s_2, \ldots, s_5', s_6', s_7'$, of that paper, page 259, correspond to ab, ac, ad, de, df, dg, dh, respectively, and if we associate with ad the letter a_1 the symmetric group of degree 8 permutes the twenty-eight binary products ab, ac, ad, ..., according to the substitutions of our simply transitive primitive group of degree 28 and class 12.

To make reference easier a table of the seven primitive groups of class 15 is given below. The second group is doubly and the third triply transitive; all the others are simply transitive.

Degree	Order	Generating Substitutions
16	80	$G_1 = \{ab . cd . ef . gh . ij . kl . mn . op,\ boejc . dpknl . fhgim\}$
16	240	$G_2 = \{G_1,\ bmn . cik . deh . flo . gpj\}$
17	4080	$G_3 = \{G_2,\ aq . fk . dg . ej . il . mn . co . hp\}$
21	2520	$G_4 = \{agb . chl . dim . ejn . fko,\ ahi . blm . cpd . eqs . frt,$
		$ajk . bno . cqr . dst . euf\}$
25	7200	$G_5 = \{H,\ a_2b_1 . a_3c_1 . a_4d_1 . a_5e_1 . b_3c_2 . b_4d_2 . b_5e_2 . c_4d_3 . c_5e_3 . d_5e_4\}$
25	14400	$G_6 = \{G_5,\ a_1b_2 . a_2b_1 . c_1c_2 . d_1d_2 . e_1e_2 . a_3b_3 . a_4b_4 . a_5b_6\}$
25	14400	$G_7 = \{H,\ a_1b_2 . a_2b_1 . c_1c_2 . d_1d_2 . e_1e_2 . a_3b_3 . a_4b_4 . a_5b_5,$
		$a_1b_1b_2a_2 . c_1b_3c_2a_3 . d_1b_4d_2a_4 . e_1b_5e_2a_5 . c_4d_3 . c_5e_3 . d_5e_4\}$

In the last three groups

$$H = \{a_1a_2a_3 . b_1b_2b_3 . c_1c_2c_3 . d_1d_2d_3 . e_1e_2e_3,\quad a_1a_4a_5 . b_1b_4b_5 . c_1c_4c_5 . d_1d_4d_5 . e_1e_4e_5,$$
$$a_1b_1c_1 . a_2b_2c_2 . a_3b_3c_3 . a_4b_4c_4 . a_5b_5c_5,\quad a_1d_1e_1 . a_2d_2e_2 . a_3d_3e_3 . a_4d_4e_4 . a_5d_5e_5\}.$$

STANFORD UNIVERSITY, *September*, 1916.

* *Comptes Rendus, loc. cit.*